influence coefficients of spherical shells, ' Am Soc Mech Engrs Paper 62-WA-58 (1962)
[11] Nachbar, W , "Discontinuity stresses in pressurized thin shells of revolution, ' Lockheed Missiles and Space Div , LMSD-48483 (1957)
[12] Timoshenko, S P , *Theory of Elastic Stability* (McGraw Hill Book Co , Inc , New York (1936) Chap VII, pp 423–425
[13] Johns, R H and Orange, T W , "Theoretical elastic stress distributions arising from discontinuities and edge loads in several shell-type structures, ' NASA TR R-103 (1961)
[14] Johns, R H , Morgan, W C , and Spera, D A , "Theoretical and experimental analysis of several typical junctions in space vehicle shell structures ' ARS Preprint 2427-62 (1962)
[15] Morgan, W C and Bizon, P T , "Experimental investigation of stress distributions near abrupt change in wall thickness in thin-walled pressurized cylinders, NASA TN (to be published)

Method for Determination of Velocity Distribution in a Thin Liquid Film

S L Persson*
Flygmotor Aeroengine Company, Trollhattan, Sweden

In conjunction with an investigation of film cooling of blunted bodies, a method has been developed for the measurement of the velocity distribution in thin liquid films The mean velocity and the thickness of the film are also determined Results are presented from measurements in water films with thicknesses of the order of 0 1 to 0 9 mm It is assumed that the suggested method also may find application in different types of boundary-layer investigations

FILM cooling has been suggested and applied as a means of protecting motor parts and aerodynamic bodies in high-temperature environments Different studies of film cooling with liquids have been reported [1–4] It is obvious from the results of these investigations that knowledge of the physics of the liquid film is of great importance to the designer of a film cooling system In particular, the stability of the film has a profound influence on the coolant efficiency

A method to measure the velocity profile in a thin liquid film (0 1 to 1 mm thick) has been developed A liquid film is generated on a plane surface from a slot injector Small metal particles are mixed uniformly with the liquid before the injection The film is photographed from above with a camera with well-known exposure time The length of the traces on the picture are proportional to the velocity of the particles The picture does not, however, reveal directly whether a certain particle has been traveling at the bottom of the film, close to the solid surface, or at the top, close to the air boundary layer

To obtain the velocity distribution, the following steps are taken First, a large number of particle traces, say 100, are measured at identical conditions Since the number of traces on each photograph is limited to around 20 (to avoid colliding traces), this means that the exposures must be repeated in short intervals (1 sec) Two assumptions are then made One is that the particle density in the liquid flow is uniform, that is, each cubic centimeter of liquid contains the same number of particles A special mixer before the injection slot is used to guarantee the uniform distribution The other assumption is that the velocity profile is monotonic The last assumption limits the use of the method to certain types of films It is not, however, considered to be a serious limitation to films pertinent to film-cooling practice

Received October 31, 1963
* Head, Physics Section, Research Department

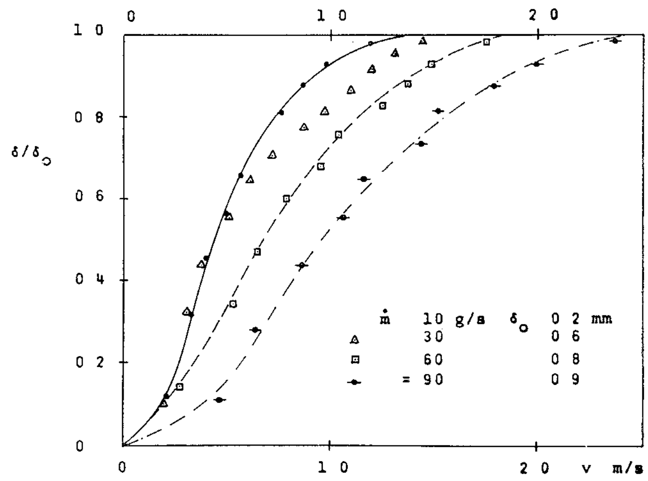

Fig 1 Velocity profiles in water film

After measurement of the traces, a diagram can be drawn over the number of particles as a function of the particle velocities Suppose that just 100 particles are measured Then, from the foregoing assumptions, it follows that the 10 particles with the highest velocity must travel in the film layer closest to the surface between air and liquid The next 10 particles will travel in the following layer, and so on The widths of these 10 layers are not equal but are proportional to the respective velocities due to the continuity principle The total film thickness can be calculated since the mass flow of liquid is known A mean velocity can also be determined

The apparatus that has been used consists of a mercury lamp for illumination of the particles and a camera The lamp was mounted so that the light rays, when hitting the particles, made as small an angle as possible with the film plane The camera, mounted perpendicular to the film plane, used an exposure time around 1 msec

In most of the experiments, nearly spherical aluminium particles with the diameters between 20 and 40 μm have been used These diameters are considered small enough to give a correct picture of the velocity profile ($d = \frac{1}{10}$ of the film thickness) Calculations have been made which show that particles that are incidentally halted at the slot have the right velocity at the test area if the particles are smaller than 0 1 mm

In Fig 1 are shown some results from the investigation Velocity profiles are given as a function of a dimensionless distance (distance from solid surface toward the interface divided by film thickness) The water mass flow was varied between 10 and 90 g/sec The film thicknesses are given All results presented in Fig 1 refer to tests with no outside air flow

Some tests have been made with an air flow blowing along the water surfaces In the tests hitherto made, no influence on the film velocity profile from the air flow has been observed No tests have been performed, however, at air velocities higher than 30 m/sec ($Re_x = 2 \times 10^5$)

The film thickness has been measured also with a separate method, using a needle that has been held close to the water surface Afterwards the distance between the needle point and the surface was measured The following measured thicknesses show the good agreement between the needle d_0 and the particle technique d:

$\dot{m}$ = 30 g/sec	50 g/sec
d = 0 37	0 56 with air (50 m/sec)
d = 0 37	0 56 without air
d_0 = 0 35	0 50 without air

Other results from the preliminary investigations are given in Ref 4

References

[1] Kinney, G , Abramson, A , and Sloop, J , Internal liquid film-cooling experiments with airstream temperatures to 2000 F in 2- and 4 inch diameter horizontal tubes," NACA Rept 1087 (1952)

[2] Polyayev, V N , "Experimental investigation of the flow of a vaporized liquid film along the surface of a cone blown by gas," Foreign Technology Div Transl FTD-TT-62-1031/1 + 2 + 4, Armed Services Tech Info Agency ASTIA AD 287701 (1962)

[3] Sellers, J P , Jr , Experimental and theoretical study of the application of film-cooling to a cylindrical rocket thrust chamber,' Ph D Thesis, Purdue Univ (1958)

[4] Persson, S L , 'Some experiments on water film-cooling, Internal Rept RF 5:1203 Flygmotor Aeroengine Co Trollhättan Sweden (1963)

Approximation of the Eigenvalues for Heat Transfer in Laminar Tube Slip Flow

ROBERT M INMAN*
NASA Lewis Research Center, Cleveland, Ohio

FOR convective heat transfer in laminar continuum tube flow with uniform wall heat flux, Sellers et al [1] have obtained asymptotic formulas for the eigenvalues and coefficients through a generalization of constant wall temperature results An improved and more direct treatment has been presented by Dzung [2]

The advent of space flight has brought about increased interest in the heat transfer to low-density gas flow in tubes Sparrow and Lin[3] have considered the fully developed heat transfer in circular tubes under slip-flow conditions It is of interest to determine the possible application of the method of Sellers et al to laminar tube slip flow

We wish to find solutions of

$$(d/d\eta)[\eta(dR_n/d\eta)] + \lambda\ (2f)\eta R = 0 \quad (1)$$

subject to

$$R_n(0) = 1 \quad (2)$$

and

$$[(dR_n/d\eta)] = 0 \text{ at } \eta = 0 \text{ and } \eta = 1 \quad (3)$$

where η is the dimensionless radial distance (r/r_0), $f(\eta)$ is the dimensionless velocity distribution,

$$f(\eta) \equiv u(\eta)/\bar{u} = 2[1 - \eta^2 + 4\alpha]/[1 + 8\alpha] \quad (4)$$

and $\alpha \equiv (\xi_u/d)$ The function R_n represents the radial temperature distribution in the thermal entrance region, λ_n is the eigenvalue, and ξ_u is the velocity slip coefficient [3] The velocity distribution, as given in Eq (4), assumes that thermal creep is negligible

In accordance with the method of Sellers et al , a solution of the form

$$R(\eta) = \exp[g(\eta)] \quad (5)$$

is considered, where

$$g = \lambda^{1/2}g_0 + g_1 + (g_2/\lambda^{1/2}) + \quad (6)$$

and, since λ is assumed to be large, only the first two terms of the foregoing series are retained It can be shown that R is given from Eqs (5) and (6) as

$$R = \left\{A \exp\left[i(\lambda)^{1/2} \int_0^\eta (2f)^{1/2}d\eta\right] + B \exp\left[-i(\lambda)^{1/2} \int_0^\eta (2f)^{1/2}d\eta\right]\right\} \Big/ \eta^{1/2}(2f)^{1/4} \quad (7)$$

excluding the singular point $\eta = 0$ It should be noted that, for continuum flow ($\alpha = 0$), a singularity also exists at $\eta = 1$, since $[f(1)]_{\alpha=0} = 0$ This has required the development of an alternate solution valid near $\eta = 1$ [1 2] For slip flow, no singularity exists at $\eta = 1$, since $[f(1)]_{\alpha\neq 0} = 8\alpha/(1 + 8\alpha) = u /\bar{u}$, where u is the slip velocity

The coefficients A and B are determined from continuation of Eq (7) to the central zone $\eta \approx 0$, where $R(\eta)$ can be approximated by a Bessel function

$$R(\eta) \approx J_0\{[2\lambda f(0)]^{1/2}\eta\} \qquad \eta^2 \ll 1 \quad (8)$$

where $f(0) = 2(1 + 4\alpha)/(1 + 8\alpha) = u /\bar{u}$ From the asymptotic expression of Bessel functions the coefficients are determined so that

$$R(\eta) = (\pi\eta)^{-1/2}(2/\lambda f)^{1/4} \cos[\lambda^{1/2}I - \pi/4] \quad (9)$$

where

$$I \equiv \int_0^\eta (2f)^{1/2}d\eta = \frac{\eta[1 + 4\alpha - \eta^2]^{1/2} + (1 + 4\alpha) \arcsin[\eta/(1 + 4\alpha)^{1/2}]}{(1 + 8\alpha)^{1/2}} \quad (10)$$

The slope of $R(\eta)$ at the wall is found by differentiating Eq (9) and setting $\eta = 1$; the result is

$$R'(1) = (8\pi)^{-1/2}(1 + 8\alpha)^{1/4}(4\alpha)^{-5/4}\lambda^{-1/4} [(E + F\gamma) \cos\gamma + (E - F\gamma) \sin\gamma] \quad (11)$$

where

$$\gamma \equiv \lambda^{1/2}I_1 \qquad I_1 \equiv \int_0^1 (2f)^{1/2}d\eta \qquad E \equiv 1 - 4\alpha$$

$$F \equiv 4(4\alpha)^{3/2}/\{(4\alpha)^{1/2} + (1 + 4\alpha) \arcsin[1/(1 + 4\alpha)^{1/2}]\}$$

Setting $R'(1) = 0$ yields a series of eigenvalues λ_n with the corresponding eigenfunction R_n as roots of the equation

$$\tan\gamma_n = (F\gamma_n + E)/(F\gamma_n - E) \quad (12)$$

The coefficients C_n of the series expansion for uniform wall heat flux are determined by the requirement that [3 4]

$$\sum_{n=1}^{\infty} C R_n(\eta) = -\left[\left(\eta^2 - \frac{1}{4}\eta^4 - \frac{7}{24}\right) - \left(\frac{1}{2}\eta^2 - \frac{1}{4}\eta^4\right)\left(\frac{u_s}{\bar{u}}\right) + \left(\frac{1}{24}\right)\left(\frac{u_s}{\bar{u}}\right)^2\right] \quad (13)$$

which, with the orthogonality property of the eigenfunctions, leads to

$$C_n = 1/[\lambda(\partial^2R/\partial\eta\partial\lambda)]_{\eta=1\ \lambda=\lambda} \quad (14)$$

Hence,

$$D_n \equiv C_nR\ (1) = -(16\alpha)/[E + (E^2/F) + F\gamma_n^2] \quad (15)$$

These expressions were derived on the assumption that λ_n is large and, consequently, are supposedly valid only in that limit However, the use of Eqs (12) and (15) gives values that appear to "fit" between results for continuum flow ($u_s = 0$) and for slug flow ($u_s/\bar{u} \to 1$) even for the values of n as small as 2, as can be seen from the comparison shown in Table 1 The results for continuum flow were obtained from expressions presented by Dzung,[2] whereas for slug flow the eigenvalues are obtained as the roots of $J_1[(2\lambda_n)^{1/2}] = 0$, where J_1 is a Bessel function of the first kind and of first order; the coefficients D_n are then obtained from the simple result $D_n = -1/\lambda_n$

Received November 6, 1963

* Aerospace Engineer Associate Member AIAA